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An application of two-edge coloured graphs to group algebras of non-noetherian groups
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AUTHOR(S):

Nishinaka, Tsunekazu; Alexander, James

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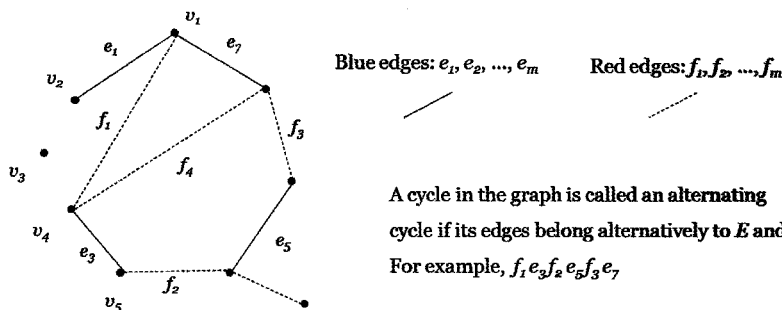
Tsunekazu Nishinaka *
University of Hyogo
nishinaka@econ.u-hyogo.ac.jp
James Alexander
University of Delaware

In this note, we introduce an SR-graph and an SR-cycle; we show that certain SR-graphs have SR-cycles. The class of SR-graphs is a subclass of the class of two-edge coloured graphs in which an SR-cycle is called an alternating cycle. We also consider an application of SR-graphs to group algebras; how to prove primitivity of group algebras of non-noetherian groups.

1 Two-edge coloured graphs

Let $\mathcal{G} = (V, E)$ be a simple graph (i.e., an undirected graph without loops or multi-edges) with vertex set V and edge set E . \mathcal{G} is a two-edge coloured graph if each of the edges is coloured either red or blue. We call a path alternating if the successive edges in \mathcal{G} alternate in colour. For any $W \subseteq V$, we let $\mathcal{G}[W]$ denote the subgraph of \mathcal{G} induced by W , i.e., $\mathcal{G}[W] := (W, \{vw \in E \mid v, w \in W\})$; let $\mathcal{G}_v := \mathcal{G}[V \setminus \{v\}]$.

A two-edge coloured graph



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We let $X(\mathcal{G})$ denote the set of all cut-vertices of \mathcal{G} , i.e., the set of all $v \in V$ so that $c(\mathcal{G}_v) > c(\mathcal{G})$. For any terminology and notation which we do not define, we follow [1] (which can also serve as an introductory text if needed).

The following result is due to Grossman and Häggkvist [3]:

Theorem 1.1. ([3, Theorem]) *Let \mathcal{G} be a two-edge coloured graph so that every vertex is incident with at least one edge of each colour. Then either \mathcal{G} has a cut vertex separating colours, or \mathcal{G} has an alternating cycle.*

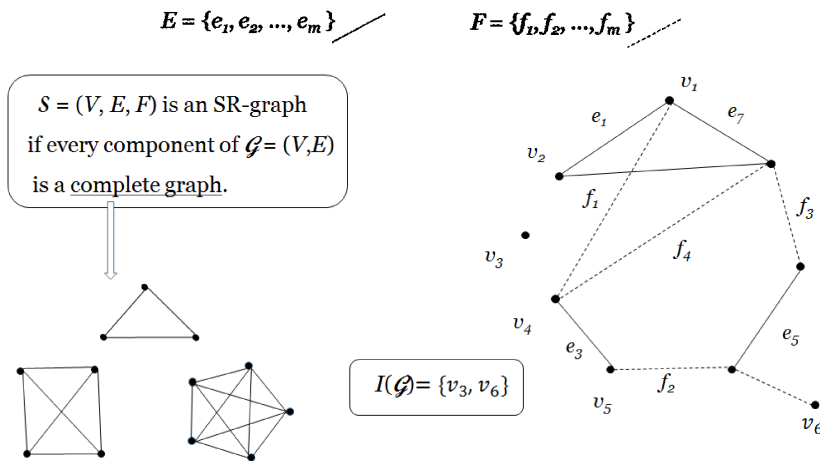
2 SR-graphs

In this section, we define an SR-graph and an SR-cycle; we show that certain SR-graphs have SR-cycles. We write $\mathcal{G} = (V, E)$ to denote that \mathcal{G} is a simple graph (undirected and without loops or multi-edges) having vertex set V and edge set E . We denote $\{v, w\} \in E$ by vw when there is no risk of confusion. We let $I(\mathcal{G})$ denote the isolated vertices of \mathcal{G} , i.e., the set of all $v \in V$ for which $vw \notin E$ for all $w \in V$. We denote by $C(\mathcal{G})$ the set of components of \mathcal{G} , i.e., the set of subgraphs of \mathcal{G} which partition \mathcal{G} , so that in each subgraph any two vertices are joined by a path, and so that no vertices which do not lie in the same subgraph are joined by a path in \mathcal{G} ; we let $c(\mathcal{G}) := |C(\mathcal{G})|$. We say that \mathcal{G} is connected if $c(\mathcal{G}) = 1$. We begin with two definitions:

Definition 2.1. Let $\mathcal{G} := (V, E)$ and $\mathcal{H} := (V, F)$. If every component of \mathcal{G} is a complete graph, and if $E \cap F = \emptyset$, then we call the triple $\mathcal{S} = (V, E, F)$ a *sprint relay graph*, abbreviated SR-graph. We view \mathcal{S} as the graph $(V, E \cup F)$, guaranteed simple as $E \cap F = \emptyset$, with edges partitioned into E and F ; we denote \mathcal{S} by $(\mathcal{G}, \mathcal{H})$ rather than (V, E, F) when convenient.

Definition 2.2. A cycle in an SR-graph (V, E, F) is called an SR-cycle if its edges belong alternatively to E and not to E ; more formally, we call cycle (V', E') an SR-cycle if there is labeling $V' = \{v_1, v_2, \dots, v_c\}$ and $E' = \{v_1v_2, v_2v_3, \dots, v_{c-1}v_c, v_cv_1\}$ so that $v_iv_{i+1} \in E$ if and only if i is odd, for some even c .

An SR-graph



The class of SR-graphs is a subclass of the class of two-edge coloured graphs in which an SR-cycle is simply an alternating cycle (see the previous section).

For the remainder of this section, fix $\mathcal{S} = (V, E, F)$, $\mathcal{G} = (V, E)$, and $\mathcal{H} = (V, F)$ so that $V \neq \emptyset$, every component of \mathcal{G} complete, and \mathcal{S} an SR-graph. Moreover, let $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$ denote the components of \mathcal{H} with $\mathcal{H}_i = (V_i, E_i)$ over $i \in [n]$. We first address the case in which \mathcal{H}_i is a complete graph for each $i \in [n]$ as follows:

Theorem 2.3. ([4, Theorem 2.3]) *If \mathcal{S} is connected and each component of \mathcal{H} is complete, then \mathcal{S} has an SR-cycle if and only if $c(\mathcal{G}) + c(\mathcal{H}) < |V| + 1$.*

Recall that $X(\mathcal{G})$ denote the set of all cut-vertices of \mathcal{G} . The

following result follows from Theorem 1.1:

Lemma 2.4. *If \mathcal{S} has no SR-cycle, then $I(\mathcal{G}) \cup I(\mathcal{H}) \cup X(\mathcal{S}) \neq \emptyset$.*

Before moving on, let us collect some straightforward observations:

Remark 2.5. Assume that \mathcal{S} , \mathcal{G} , and \mathcal{H} satisfy the hypotheses of Theorem 2.3.

(I) If $v \notin X(\mathcal{S})$, then

(i) $v \in I(\mathcal{G}) \cup I(\mathcal{H})$ implies $c(\mathcal{G}_v) + c(\mathcal{H}_v) = c(\mathcal{G}) + c(\mathcal{H}) - 1$;

(ii) $v \notin I(\mathcal{G}) \cup I(\mathcal{H})$ implies $c(\mathcal{G}_v) = c(\mathcal{G})$ and $c(\mathcal{H}_v) = c(\mathcal{H})$.

(II) If $v \in X(\mathcal{S})$, then without loss of generality,

(i) \mathcal{S}_v is an SR-graph with components $(\mathcal{G}_1, \mathcal{H}_1)$ and $(\mathcal{G}_2, \mathcal{H}_2)$;

(ii) $\sum_{i=1}^2 (c(\mathcal{G}_i) + c(\mathcal{H}_i)) = c(\mathcal{G}) + c(\mathcal{H})$ and $|V_1| + |V_2| = |V| - 1$, where V_1 and V_2 are the vertex sets of $(\mathcal{G}_1, \mathcal{H}_1)$ and $(\mathcal{G}_2, \mathcal{H}_2)$, respectively.

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. Before entering the heart of this proof, we show that

$$c(\mathcal{G}) + c(\mathcal{H}) \leq |V| + 1, \quad (1)$$

which holds trivially when $|V| = 1$. Assume, by way of induction, that $|V| > 1$ and that (1) holds for SR-graphs on fewer vertices. Fix $v \in V$. If $v \notin X(\mathcal{S})$, then \mathcal{S}_v is connected and \mathcal{H}_v has complete components; thus, $c(\mathcal{G}_v) + c(\mathcal{H}_v) \leq |V|$ by induction, and so (1) follows from Remark 2.5(I). If $v \in X(\mathcal{S})$, then \mathcal{S}_v has components $(\mathcal{G}_1, \mathcal{H}_1)$ and $(\mathcal{G}_2, \mathcal{H}_2)$ by Remark 2.5(II)(i); by induction, $c(\mathcal{G}_i) + c(\mathcal{H}_i) \leq |V_i| + 1$ for $i \in [2]$, and thus (1) holds by Remark 2.5(II)(ii).

We are now ready for the crux of our argument. First, assume that \mathcal{S} has an SR-cycle. We prove by induction on $|V|$ that $c(\mathcal{G}) + c(\mathcal{H}) < |V| + 1$, noting that we may assume $|V| \geq 4$. This holds trivially if $|V| = 4$, so assume $|V| > 4$ and, by way of induction, that the result holds for SR-graphs on fewer vertices. This result holds trivially if \mathcal{S} is an SR-cycle, so we may assume that there is $C \subsetneq V$ so that $\mathcal{S}[C]$ is an SR-cycle.

Consider $v \in V \setminus C$. If $v \notin X(\mathcal{S})$, then we can obtain the desired result with a similar argument to that which we used in the first paragraph when $v \notin X(\mathcal{S})$ was assumed. Assume $v \in X(\mathcal{S})$, in which case \mathcal{S}_v has components $(\mathcal{G}_1, \mathcal{H}_1)$ and $(\mathcal{G}_2, \mathcal{H}_2)$ by Remark 2.5(II)(i). Since $v \in X(\mathcal{S})$ and \mathcal{G} and \mathcal{H} have complete components, either $C \subseteq V_1$ or $C \subseteq V_2$; say, without loss of generality, that $C \subseteq V_1$. Then, by our induction hypothesis, $c(\mathcal{G}_1) + c(\mathcal{H}_1) < |V_1| + 1$. Also, by (1), $c(\mathcal{G}_2) + c(\mathcal{H}_2) \leq |V_2| + 1$. Thus, by Remark 2.5(II)(ii) that $c(\mathcal{G}) + c(\mathcal{H}) < |V| + 1$.

To prove the converse, by (1), it suffices to show that if \mathcal{S} has no SR-cycle, then $c(\mathcal{G}) + c(\mathcal{H}) = |V| + 1$. To that end, assume \mathcal{S} has no SR-cycle. Our proof will again be by induction on $|V|$. If $X(\mathcal{S}) \neq \emptyset$ then we may consider $v \in X(\mathcal{S})$ and obtain the result with a similar argument to that which we used in the first paragraph when $v \in X(\mathcal{S})$ was assumed. Assume $X(\mathcal{S}) = \emptyset$. By Lemma 2.4, there is $v \in I(\mathcal{G}) \cup I(\mathcal{H})$. By induction, $c(\mathcal{G}_v) + c(\mathcal{H}_v) = |V|$. It follows from Remark 2.5(I)(i) that $c(\mathcal{G}) + c(\mathcal{H}) = |V| + 1$. \square

Let $I := I(\mathcal{G})$, $W := V \setminus I$, $W_i := V_i \setminus I$, and say $\mathcal{H}[W_i] = (W_i, F_i)$. For any $m_1, m_2, \dots, m_k \in \mathbb{N}$, we let K_{m_1, m_2, \dots, m_k} denote the complete multipartite graph with partite sets of size m_1, m_2, \dots, m_k , i.e., the graph (V', E') so that V' can be partitioned into sets P_1, P_2, \dots, P_k called partite sets, with $|P_i| = m_i$ and $vw \in E'$ if and only if v and w are in different partite sets for all $v, w \in V$. We let $\mu(K_{m_1, m_2, \dots, m_k}) := \max_{i \in [k]} \{m_i\}$. We now handle the case in which each component of

\mathcal{H} is complete multipartite. We can then get the following theorem:

Theorem 2.6. ([4, Theorem 2.6]) *Assume that \mathcal{H}_i is a complete multipartite graph for each $i \in [n]$. If $|I| \leq n$ and $|V_i| > 2\mu(\mathcal{H}_i)$ for each $i \in [n]$, then \mathcal{S} has an SR-cycle.*

In order to build to a proof of Theorem 2.6, we need two lemmas (see [4]).

Lemma 2.7. *Let $U \subseteq V$ with $U \cap I = \emptyset$, and let $U' := V \setminus U$. Then, $|I \cap U'| \leq |I(\mathcal{G}[U'])| \leq |I \cap U'| + |U|$.*

Lemma 2.8. *If $\mathcal{H}[W_i] \not\simeq K_{1,m}$ for all $m \geq 2$ and $I(\mathcal{H}[W]) = \emptyset$, then \mathcal{S} has an SR-cycle.*

We are now ready to prove Theorem 2.6.

Proof of Theorem 2.6. Our proof is by induction on n . Assume $n = 1$, and say \mathcal{H}_1 has partite sets P_1, P_2, \dots, P_p . We note that if there are distinct $i, j \in [p]$, and $v_i, w_i \in P_i$ and $v_j, w_j \in P_j$ with $v_i w_i, v_j w_j \in E$, then $\mathcal{S}[\{v_i, w_i, v_j, w_j\}]$ is an SR-cycle by definition. So, we may assume, without loss of generality, that elements of E join only vertices of P_1 (and thus, that $P_i \subseteq I$ for $i \neq 1$). However, as $|V_1| > 2|P_1|$, this implies that $|I| \geq |V_1 \setminus P_1| > 1$, so this case cannot occur, and thus the desired result holds when $n = 1$. Assume, by way of induction, that this result holds for all SR-graphs (V', E', F') satisfying analogous hypotheses, if (V', F') has less than n components.

Suppose that there is $i \in [n]$ with $\mathcal{H}[W_i] \simeq K_{1,m}$ for some $m \geq 2$. Since $|W_i| = |V_i| - |I \cap V_i|$ by definition, and since $|W_i| = m + 1$ by assumption, it follows from our hypotheses that

$$m + 1 > 2\mu(\mathcal{H}_i) - |I \cap V_i| \geq 2m - |I \cap V_i|, \quad (2)$$

since $\mu(\mathcal{H}_i) \geq \mu(\mathcal{H}[W_i]) = m$. Let P_1, P_2, \dots, P_k be the partite sets of \mathcal{H}_i , and let $Q_1 = \{w_0\}$ and $Q_2 = \{w_1, w_2, \dots, w_m\}$ be the partite sets of $\mathcal{H}[W_i]$; without loss of generality, say $Q_1 \subseteq P_1$ and $Q_2 \subseteq P_2$. Now, since $|V_i| > 2\mu(\mathcal{H}_i)$, $k \geq 3$; since $\mathcal{H}[W_i] \simeq K_{1,m}$, this implies that there is $v \in P_3 \cap I$. Let V' be obtained from V by replacing V_i with $V'_i := \{w_0, w_1, v\}$, and consider $\mathcal{S}[V']$. Since $\mathcal{H}[V'_i] \simeq K_{1,1,1}$, we have $|V'_i| > 2\mu(\mathcal{H}[V'_i])$. Moreover, if the vertices in $Q_2 \setminus \{w_1\}$ are removed from V , then the number of additional isolated vertices caused by the removing of those vertices is at most $|Q_2 \setminus \{w_1\}|$ by Lemma 2.7. Moreover $|(I \cap V_i)| \geq m$ by (2), and so it holds that

$$\begin{aligned} |I(\mathcal{G}[V'])| &\leq |I| - |(I \cap V_i) \setminus \{v\}| + |Q_2 \setminus \{w_1\}| \\ &\leq n - (m - 1) + (m - 1) = n. \end{aligned}$$

Therefore, $\mathcal{S}[V']$ still satisfies the hypotheses of our theorem, and clearly, if $\mathcal{S}[V']$ has an SR-cycle then so must \mathcal{S} . Moreover, by considering corresponding $W'_i = \{w_0, w_1\}$, we see that $\mathcal{H}[W'_i] \simeq K_{1,1}$ (and, in particular, no longer isomorphic to $K_{1,m}$ for any $m \geq 2$). Thus, we may assume that $\mathcal{H}[W_i] \not\simeq K_{1,m}$ (by applying this procedure to any component of \mathcal{H} if necessary).

Since $\mathcal{H}[W_i] \not\simeq K_{1,m}$ for any $m \geq 2$, if $F_i \neq \emptyset$ for all $i \in [n]$ (as this is equivalent to $I(\mathcal{H}[W]) = \emptyset$ in this case), then we obtain the desired result by Lemma 2.8. So, it remains to assume that $\mathcal{H}[W_i] \not\simeq K_{1,m}$, but that $F_i = \emptyset$ for some i . Let $V' := V \setminus V_i$ and say $\mathcal{S}[V'] = (V', E', F')$. Since the number of components of (V', F') is $n - 1$, we may apply our induction hypothesis and prove this result if $|I(\mathcal{G}[V'])| \leq n - 1$; we show that this must be the case. Let $m := |W_i|$. Since \mathcal{H}_i is a complete k -partite graph and $F_i = \emptyset$, W_i is contained in a partition of \mathcal{H}_i , and so $|V_i| > 2m$ by assumption; thus, $|I \cap V_i| = |V_i| - m > m$. Since $I \cap V' = I \setminus (I \cap V_i)$ and $|I| \leq n$, we have $|I \cap V'| \leq n - m - 1$. On the other hand, by Lemma 2.7, $|I(\mathcal{G}[V'])| - |I \cap V'| \leq m$. Hence,

$$m \geq |I(\mathcal{G}[V'])| - |I \cap V'| \geq |I(\mathcal{G}[V'])| - (n - m - 1),$$

and thus $|I(\mathcal{G}[V'])| \leq n - 1$. □

3 How to apply SR-graph theory to algebras

In order to prove the group algebra $R = KG$ of a group G over a field K to be primitive, according to the method of Formanek [2], it suffices to show that for each non-zero $a \in R$, there exists an element $\varepsilon(a)$ in the ideal RaR generated by a such that the right ideal $\rho = \sum_{a \in R \setminus \{0\}} (\varepsilon(a) + 1)R$ is proper. The main difficulty here is how to choose elements $\varepsilon(a)$'s so as to make ρ be proper. Now, ρ is proper if and only if $r \neq 1$ for all $r \in \rho$. Since ρ is generated by the elements of form $(\varepsilon(a) + 1)b$ with $a \neq 0$, r has the presentation, $r = \sum_{(a,b) \in \Pi} (\varepsilon(a) + 1)b$, where Π is a subset of $R \times R$ consisting of a finite number of elements both of whose components are non-zero. Moreover, since $\varepsilon(a)$ and b are linear combinations of elements of G , r is presented as follows:

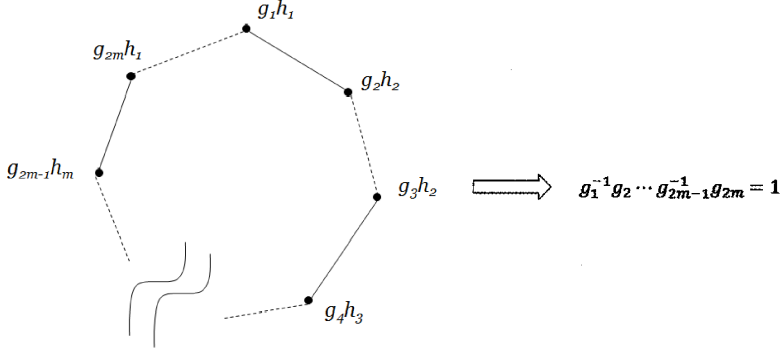
$$r = \sum_{(a,b) \in \Pi} \sum_{g \in S_a, h \in T_b} (\alpha_g \beta_h gh + \beta_h h), \quad (3)$$

where S_a and T_b are the support of $\varepsilon(a)$ and b respectively and both α_g and β_h are elements in K . In the above presentation (3), if there exists gh such that $gh \neq 1$ and does not coincide with the other gh 's and h 's, then $r \neq 1$ holds.

On the contrary, if $r = 1$, then for each gh in (3) with $gh \neq 1$, there exists another $g'h'$ or h' in (3) such that either $gh = g'h'$ or $gh = h'$ holds. Suppose here that there exist (g_{2i-1}, h_i) and (g_{2i}, h_{i+1}) ($i = 1, \dots, m$) in $V = \bigcup_{(a,b) \in \Pi} S_a \times T_b$ such that the

following equations hold:

$$\begin{aligned}
 g_1 h_1 &= g_2 h_2, \\
 g_3 h_2 &= g_4 h_3, \\
 &\vdots \\
 g_{2m-1} h_m &= g_{2m} h_{m+1} \quad \text{and} \quad h_{m+1} = h_1.
 \end{aligned} \tag{4}$$



Eliminating h_i 's in the above, we can see that (4) above implies the equation $g_1^{-1} g_2 \cdots g_{2m-1} g_{2m} = 1$. If we can choose $\varepsilon(a)$'s so that their supports g_i 's never satisfy such an equation, then we can prove that $r \neq 1$ holds by contradiction. We need therefore only to see when supports g 's of $\varepsilon(a)$'s satisfy equations as described in (4) provided $r = 1$ holds.

In order to see this, we consider a graph which has two distinct edge sets E and F on the same vertex set V ; an SR-graph $\mathcal{S} = (V, E, F)$. Roughly speaking, we regard $V = \bigcup_{(a,b) \in \Pi} S_a \times T_b$ above as the set of vertices and for $v = (g, h)$ and $w = (g', h')$ in V , we take an element vw as an edge in E provided $gh = g'h'$ in G , and take vw as an edge in F provided $g \neq g'$ and $h = h'$ in G . In this situation, if there exists an SR-cycle $v_1 w_1 v_2 w_2, \dots, v_p w_p v_1$ in the SR-graph (V, E, F) , then there exist (g_i, h_j) 's in V satisfying the desired equations as described in (4). Thus the problem can be reduced to find an SR-cycle in a given SR-graph.

In fact, by making use of the method described above, we can show primitivity of group algebras of groups which belong to many classes of non-noetherian groups, including free groups, locally free groups, free products, amalgamated free products, HNN-extensions and one relator groups.

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